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2010 J. Phys. A: Math. Theor. 43 215203

(http://iopscience.iop.org/1751-8121/43/21/215203)

J. Phys. A: Math. Theor. **43** (2010) 215203 (16pp) doi:10.1088/1751-8113/43/21/215203

# Autoresonance in a dissipative system

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Received 2 December 2009, in final form 31 March 2010
Published 7 May 2010
Online at stacks.iop.org/JPhysA/43/215203

**Abstract**
We study the autoresonant solution of Duffing's equation in the presence of dissipation. This solution is proved to be an attracting set. We evaluate the maximal amplitude of the autoresonant solution and the time of transition from autoresonant growth of the amplitude to the mode of fast oscillations. Analytical results are illustrated by numerical simulations.

PACS numbers: 02.30.Gp, 02.30.Hq

## Introduction

The term autoresonance refers to the growth of the amplitude of oscillations of a solution to a nonlinear equation under the action of an external oscillating force. This phenomenon looks like phase locking of a nonlinear oscillator through a periodic driver. The phase locking was first suggested to accelerate relativistic particles, see [1, 2]. Nowadays, the autoresonance is thought of as a universal phenomenon which occurs in a wide range of oscillating physical systems from astronomical to atomic ones [3].

In the general case, the frequency of a nonlinear oscillator depends on the amplitude of oscillations or, what is the same, on their energy. Hence, in order to change the energy of a nonlinear oscillator, the frequency of the external force should be adapted to that of the oscillator. If the force is small then the energy of the oscillations changes slowly. In order to remain resonant, the frequency of the external force should also adapt itself slowly to the frequency of the nonlinear oscillator. Moreover, the backward phenomenon proves to occur. Namely, the slow change of the frequency of the driver results in that the frequency of the nonlinear oscillator actually follows the driver frequency. For a contemporary survey of the mathematical aspects of autoresonance, we refer the reader to [4].

The autoresonance phenomenon in systems with dissipation was earlier studied both by means of mathematical models and in physical experiments. In particular, the existence of an autoresonant solution for the system of three coupled oscillators with small dissipation was established in [5] and for the system with parametric autoresonance in [6]. In the papers [7, 8], the threshold of capture into autoresonance was discussed in the presence of dissipation. The resonant phase-locking phenomenon in van der Pol Duffing's equation with an external driver of slowly varying frequency was studied in [9].

In this paper we treat two problems for autoresonance in dissipative systems as open for the time being. Firstly we prove the existence of an attracting set for solution trajectories captured into autoresonance. Such an attracting set was observed numerically in a number of papers [5, 6, 10, 11]. The attractor in these systems is a slowly varying steady-state solution. The solutions captured into autoresonance oscillate around such a solution and lose the energy of oscillations because of dissipation. Therefore, all captured solutions tend to the steady-state solution. Mathematically this means that the slowly varying steady-state solution is Lyapunov stable.

The second problem we deal with consists in evaluating the bound of the autoresonant growth of solution in the presence of small dissipation in the system. The earlier one observed that the amplitude growth of nonlinear oscillations in systems with dissipation is bounded, see [10–12]. From the physical viewpoint, the boundedness of autoresonant growth can be easily explained. Namely, the work of the driver is proportional to the length of the trajectory in the phase space. If the dissipation depends linearly on the velocity, then its work is proportional to the area described by the phase trajectory. Under the growth of energy, the area of the phase curve grows faster than its length. It follows that even if the dissipation is small, its work exceeds the work of the external force at some moment and the autoresonant growth of solution stops. Mathematically this looks like the impossibility of extension of the solution under phase capture. What happens is the hard loss of stability and passage to fast oscillations.

This work is aimed at finding an asymptotic expansion for the slowly varying steady-state solution to the primary resonance equation and at showing that it is an attracting set for those solutions which are captured into the resonance. Moreover, we derive asymptotics for the maximal amplitude of oscillations under autoresonance with small dissipation and calculate the period of the autoresonant mode in the solution.

The paper contains seven sections. In section 1 we describe the mathematical setting of the problem. Section 2 provides a detailed exposition of the main results. Section 3 deals with asymptotics for the autoresonant mode. In section 4 we discuss the stability of autoresonant growth. In section 5 we study the solution behaviour in the vicinity of the break of autoresonant growth. In section 6 we will look more closely at the break of autoresonant growth. Section 7 is concerned with passage from monotone autoresonant growth of the amplitude of nonlinear oscillations to fast motion solutions.

## 1. Setting of the problem

We study solution of the primary resonance equation

$$\imath\Psi' + (T - |\Psi|^2)\Psi + \imath\delta\Psi = f, \tag{1.1}$$

where $T$ is an independent variable, $\delta$ a dissipation parameter and $f$ is a parameter related to the amplitude of external force.

The primary resonance equation is of universal character in the mathematical description of autoresonance. In the case $\delta = 0$, it was first introduced in the paper [13].

The primary resonance equation describes long-term evolution of nonlinear oscillations under action of a small external force. As but one example we mention Duffing's equation with dissipation

$$u'' + u + bu' - cu^3 = \varepsilon A\cos(\omega t), \tag{1.2}$$

where $c$ and $A$ are constants, and $b$ and $\varepsilon$ small positive parameters. The frequency of oscillations on the right-hand side of the equation depends linearly on the time. More precisely, $\omega = 1 - \alpha t$ which is usually referred to as a chirped frequency. The parameter $\alpha$ (called a chirp rate) determines the rate of change of the frequency.

Duffing's equation (1.2) proves to be the simplest and so the most general equation where one observes the phenomenon of autoresonance break because of small dissipation.

For studying autoresonance, it is convenient to use the method of two scales. The oscillations of the nonlinear equation are observed in the time scale $t$. The amplitude of these oscillations depends in turn on the slow time $T := \varepsilon^{2/3}t$.

The introduction of two time scales enables one to split the evolution of solution into two parts, fast and slow ones, using the asymptotic substitution

$$u \sim \varepsilon^{1/3}\Psi(T)\,\mathrm{e}^{\imath(t-T^2)} + \text{complex conjugate term}$$

in equation (1.2). The standard averaging procedure over the fast time $t$ in the leading-order term in $\varepsilon$ leads to equation (1.1) for the unknown function $\Psi$, where $\delta = \varepsilon^{-2/3}b/4$, $\alpha = \varepsilon^{4/3}$, $f = A/(4\sqrt{2})$ and $c = -2\sqrt{2}$.

This primary resonance equation is often written as the system of equations for the amplitude $R(T) = |\Psi(T)|$ and the phase $\varphi(T) = \arg(\Psi)$. More precisely,

$$\begin{aligned} R' &= -\delta R - f\sin\varphi, \\ \varphi' &= (T - R^2) - \frac{f}{R}\cos\varphi, \end{aligned} \tag{1.3}$$

cf [14].

The autoresonance or phase locking for the solution of system (1.3) means that $\varphi' = o(1)$ for $T \to \infty$. This condition along with the second equation of system (1.3) determines the behaviour for the amplitude growth which reads $R = \sqrt{T} + o(1)$. The first equation of (1.3) gives a sufficient condition for the instant at which the phase locking is destroyed, namely $T_* = f^2/\delta^2$. The analysis of the equation and phase locking condition actually yields an estimate of autoresonance growth in a dissipative system with small dissipation, $\delta \ll 1$, see [10–12].

In this paper, we construct asymptotics for the slowly varying steady-state solution of equation (1.3) with $\delta \ll 1$. We prove that this solution is an attracting set for the captured solutions. Moreover, we show asymptotics of the maximal value of $R$ and evaluate the instant at which the phase locking is destroyed.

In order to better motivate the problem, we demonstrate results of numerical simulations for equation (1.1) with $\delta > 0$. In figure 1 one can observe three stages of evolution for the solution of (1.1). At the first stage, the oscillations are close to some smooth curve. Then, at the second stage the solution varies slowly. Finally, at the third stage the solution loses its stability and the amplitude of fast oscillations tends to zero.

## 2. Results of the paper

To formulate the results it is convenient to change both the independent and dependent variables by

$$\begin{aligned} \theta &= T\delta^2, \\ \delta\Psi(T) &= \psi(\theta, \delta). \end{aligned} \tag{2.1}$$

The equation for $\psi$ takes the form

$$\imath\delta^4\psi' + (\theta - |\psi|^2)\psi + \imath\delta^3\psi = \delta^3 f. \tag{2.2}$$

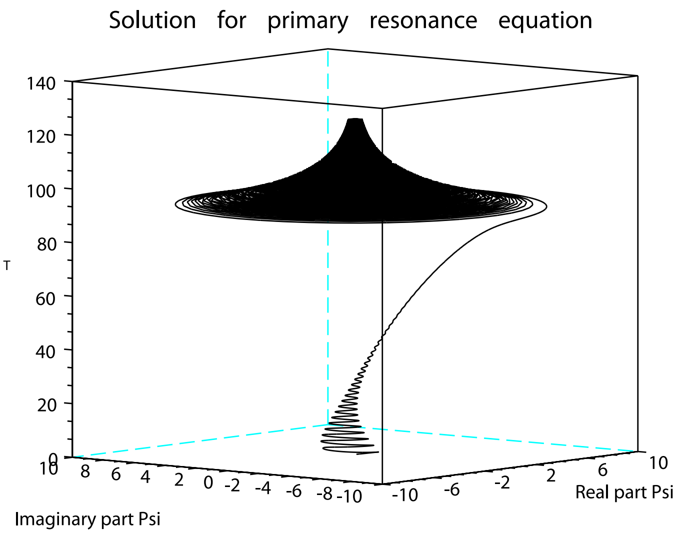


**Figure 1.** The solution of (1.1) with parameters $f = 1$, $\delta = 0.1$, and the initial condition $\Psi\restriction_{T=0} = 0$. The modulus of the solution increases slowly, and then at $T = 100$ the solution passes from slow changes to fast oscillations under amplitude decay.

Denote $R = |\psi|$ and $\varphi = \arg(\psi)$ for $\theta > 0$ and $0 < \delta \ll 1$.

In the following, we will work with the primary resonance equation in the form (2.2) and construct asymptotics of the solution to this equation for finite values of the parameter $\theta$.

The existence time for the autoresonant mode in the solution of (1.1) is evaluated by

$$\theta_* = f^2 - \delta + \delta^2\left(\sqrt[5]{\frac{6}{f^2}}z_0 - \frac{1}{4f^2}\right) + O(\sqrt{\delta^5}),$$

where $z_0 \sim 2.38$ is the first real pole of the Painlevé-1 transcendental with zero monodromy data $y_1(z, 0, 0)$. Furthermore, the maximal amplitude is estimated by

$$R_* \sim f - \frac{\delta}{2f} + \delta^2\left(\frac{1}{2f}\sqrt[5]{\frac{6}{f^2}}z_0 - \frac{1}{4f^3}\right) + O(\sqrt{\delta^5}).$$

The comparison of these asymptotics and numerical results is given in figure 2.

We are now able to give an explicit description of asymptotics for the autoresonant solution which is an attracting set for the solutions captured into autoresonance.

If $(f^2 - \theta)\delta^{-1} \gg 1$, then $R$ and $\varphi$ behave like

$$R(\theta,\delta) \sim \sqrt{\theta} + \delta^3\frac{\sqrt{f^2-\theta}}{2\theta} - \delta^4\frac{1}{2\theta\sqrt{f^2-\theta}} + \delta^5\frac{f^2-4\theta}{16\theta^2(f^2-\theta)^{3/2}} + \delta^6\frac{(\theta-3f^2)(\theta-f^2)^3 + \theta^{3/2}\sqrt{f^2-\theta}}{8\theta^{5/2}(f^2-\theta)^3}, \tag{2.3}$$

$$\varphi(\theta,\delta) \sim -\arctan\left(\frac{\sqrt{\theta}}{\sqrt{f^2-\theta}}\right) + \delta\frac{1}{2\sqrt{\theta}\sqrt{f^2-\theta}} + \delta^2\frac{1}{8\sqrt{\theta}(f^2-\theta)^{3/2}} + \delta^3\frac{(f^2+2\theta)\sqrt{f^2-\theta} - 24\sqrt{\theta}(\theta-f^2)^3}{48(\theta-f^2)^3\theta^{3/2}}. \tag{2.4}$$

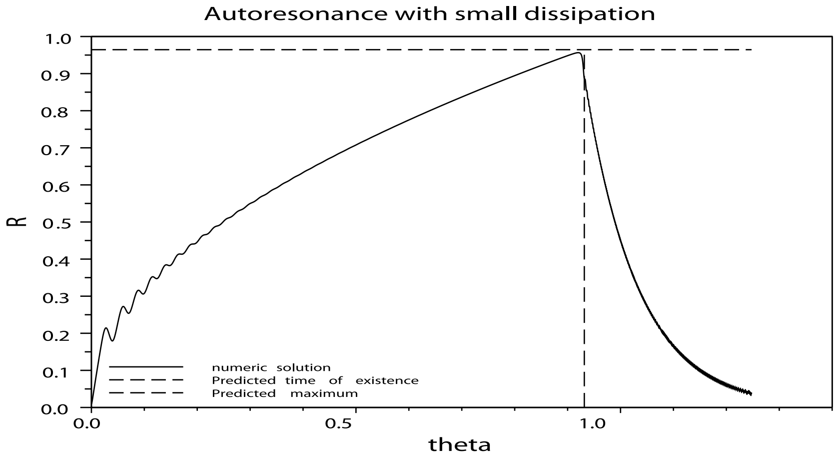


**Figure 2.** The modulus of the solution to (2.2) with parameters $f = 1$, $\delta = 0.1$ and the initial condition $\psi\restriction_{\theta=0} = 0$. Here the predicted maximum and time of existence are $R_*$ and $\theta_*$, respectively.

This asymptotics is obtained in section 3.

To write the asymptotics in a neighbourhood of $\theta = f^2$, we change the variables by

$$\eta = (\theta - f^2)\delta^{-1},$$
$$r(\eta, \delta) = (R - f)\delta^{-1},$$
$$a(\eta, \delta) = (\varphi - (3/2)\pi)\delta^{-1/2}.$$

The functions $r(\eta, \delta)$ and $a(\eta, \delta)$ have the form

$$r \sim \frac{\eta}{2f} - \delta\frac{\eta^2}{8f^3} + \delta^2\frac{\eta^3}{16f^5} - \delta^{5/2}\frac{(2\eta+3)}{4f^2\sqrt{-\eta-1}},$$
$$a \sim \frac{\sqrt{-\eta-1}}{f} - \delta\frac{(2\eta^2+4\sigma-1)}{24f^3\sqrt{-\eta-1}},$$

the representations being valid if $\delta(-1-\eta)^{-1} \ll 1$. These formulae are derived in section 5.

Close to $\eta = -1$ it is convenient to represent the asymptotics in the form

$$\tau = \frac{\eta+1}{\delta},$$
$$a = \delta^{1/2}u(\tau, \delta),$$
$$r = -\frac{1}{2f} + \delta\frac{4f^2\tau - 1}{8f^3} + \delta^2 v(\tau, \delta).$$

In this domain, the autoresonant mode of the solution loses its stability. The leading-order term of $u$ relative to $\delta$ admits the representation

$$u(\tau) \sim \sqrt[5]{\left(\frac{6}{f^2}\right)^3}\, y_1(z, 0, 0),$$
$$\tau = \sqrt[5]{\frac{6}{f^2}}z - \frac{1}{4f^2},$$

where $y_1(z,0,0)$ is the Painlevé-1 transcendental, see [18], i.e. a special solution of the Painlevé-1 equation $y_1'' = 6y_1^2 + z$ with asymptotics

$$y_1(z) = \sqrt{\frac{-z}{6}} + O(z^{-1/2}).$$

The asymptotic formula for $v$ looks like

$$v(\tau) \sim \frac{(\tau - u')}{2f}$$

as $\delta \to 0$.

The Painlevé-1 transcendental $y_1(z,0,0)$ has poles on the real axis. The approximate solution of (1.1) by means of $y_1(z,0,0)$ is valid up to a small neighbourhood of the first $z_0 \sim 2.38$ of these poles or, what is the same, up to $\tau = \tau_0 := \sqrt[5]{\frac{6}{f^2}} z_0 - \frac{1}{4f^2}$. Near the pole, the validity domain is determined by the inequality

$$\frac{\sqrt{\delta}}{\tau - \tau_0} \ll 1.$$

These asymptotic formulae for the solution are established in section 6.

The asymptotics of the solution of (1.1) in a neighbourhood of the pole represents by fast non-autoresonant oscillations in the new scale of variable $\theta = f^2 - \delta + \delta^2(\tau_0 + 3.27\delta) + \sqrt{\delta^5}\xi$. It is convenient to write the unknown functions in the form

$$R \sim f - \frac{\delta}{2f} + \delta\sqrt{\delta}\, p(\xi),$$

$$\varphi \sim \frac{3}{2}\pi + s(\xi).$$

The function $s(\xi)$ is a special solution of the equation

$$s' = -\sqrt{E + f^2(s - \sin s)},$$

such that $s \to 0-$ as $\xi \to -\infty$. The function $p(\xi)$ is determined from the equation

$$p = -\frac{s'}{2f}.$$

Note that the function $s(\xi)$ depends on a parameter $E = p^2 + (s - \sin s)$ which tends to 0 as $\xi \to -\infty$. Asymptotic formulae for the rapidly oscillating mode are obtained in section 7.

The results obtained are formulated for those solutions which are captured into autoresonance. In general, the problem of separating the domains of initial data for solutions, which are captured into autoresonance and which are not, remains open. However, numerical simulations show that for $T = 0$ there is a disc of finite radius in a neighbourhood of the origin, from which all solutions are captured into autoresonance, cf [4].

In the limit case $\delta \to 0$, the form (2.2) of equation (1.1) no longer makes sense. However, after the inverse transformations $T = \theta/\delta^2$ and $\Psi = \delta^{-1}\psi$, the asymptotic formulae (2.3) and (2.4) give a familiar asymptotics [4] for the solution of equation (1.1) when $T \to \infty$.

The important role of Painlevé's transcendentals in passage from slow changes to fast oscillations in the solutions of second order equations with slowly varying coefficients was first observed in [15]. A detailed study of reorganization from slowly varying modes to fast oscillations for the primary resonance equation without dissipation was given in [16, 17]. In these works, reorganizations are caused in the end by the non-autonomy of the primary resonance equation. The present paper deals with reorganizations which are caused by the presence of dissipation in the system. However, also in this case, the behaviour of the

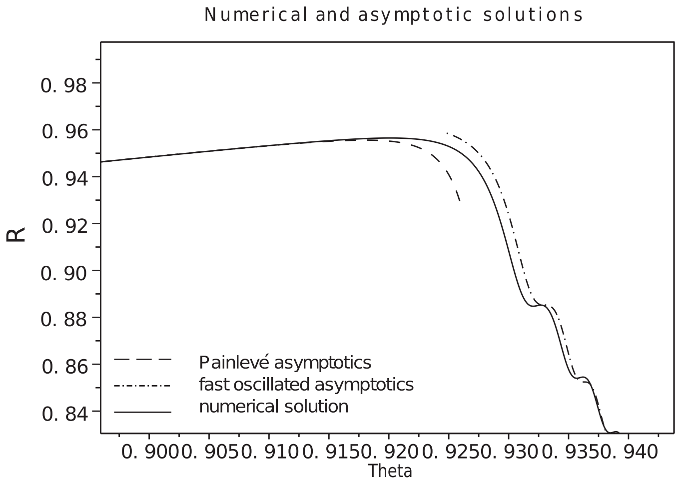


**Figure 3.** The interval of adjusting asymptotics in passage from Painlevé's layer to the layer of fast oscillations. The numerical solution of (2.2) corresponds to $f = 1$, $\delta = 0.1$ and $\psi\restriction_{\theta=0} = 0$.

solution in a neighbourhood of the reorganization is completely determined by Painlevé's transcendentals.

We now compare the asymptotics obtained with the numerical solution. For clarity we bring graphs of the amplitude of numerical solution with zero initial data at $\theta = 0$ for $\varepsilon = 0.1$ and the amplitude of asymptotic solution. The asymptotic solution is composite. We demonstrate, e.g., the region of passage to fast oscillations in figure 3.

For diverse intervals of $\theta$, the mentioned asymptotics approximates the solution with different precision. Hence it is reasonable to consider the difference between the numerical solution and asymptotics in the corresponding intervals.

The graphs given in figure 4 demonstrate rather strikingly that the difference between the numerical and asymptotic solutions increases in a neighbourhood of passage to fast oscillations. This is explained by the fact that the correction to the leading-order term in the asymptotic solutions for fast oscillations has a lower order in $\delta$, namely $\delta^2$.

## 3. Asymptotics of autoresonant growth

In this section, we construct an asymptotic solution to (2.2) in the domain $(f^2 - \theta)\delta^{-1} \gg 1$ and $\theta > 0$. To this end, we introduce new unknown functions $\rho(\theta, \delta)$ and $\alpha(\theta, \delta)$ related to the amplitude $R$ and phase $\varphi$ of the unknown function $\psi$ by

$$R(\theta, \delta) = \sqrt{\theta} + \delta^3 \rho(\theta, \delta),$$
$$\varphi(\theta, \delta) = \alpha(\theta, \delta).$$

On substituting these formulae into (2.2) and separating the real and imaginary parts of the equation, we get

$$\begin{aligned} &\delta^4 \rho' + \sqrt{\theta} + f \sin\alpha + \delta \frac{1}{2\sqrt{\theta}} + \delta^3 \rho = 0, \\ &(\delta\sqrt{\theta} + \delta^4 \rho)\alpha' + 2\theta\rho - f\cos\alpha + 3\delta^3\sqrt{\theta}\rho^2 + \delta^6\rho^3 = 0. \end{aligned} \tag{3.1}$$

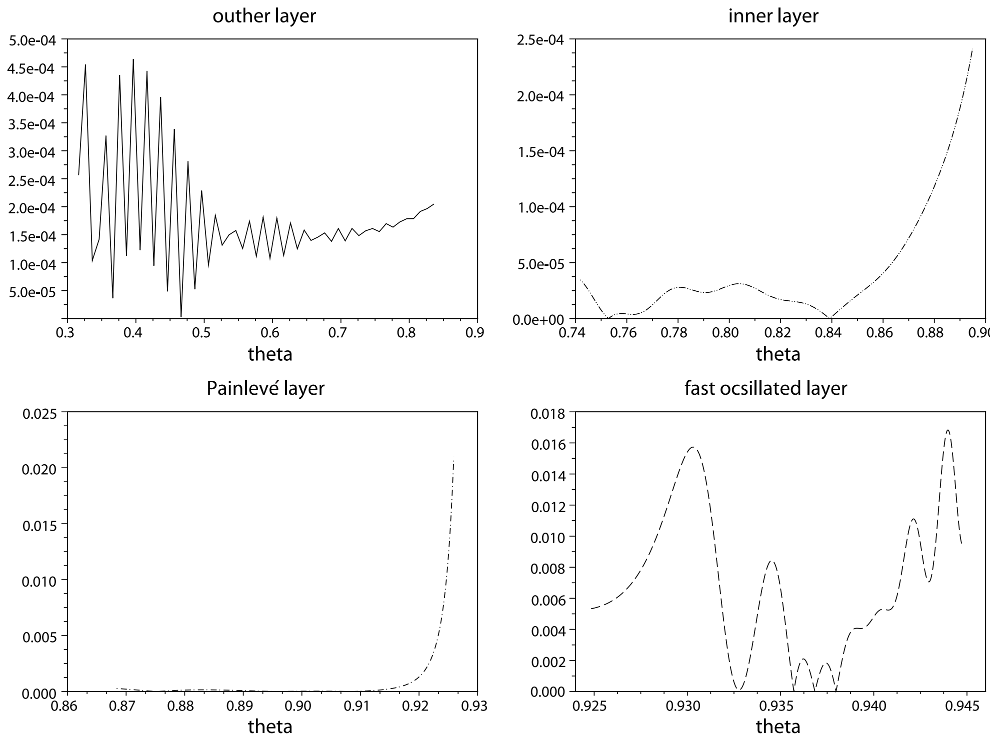


**Figure 4.** Here the absolute values of difference between the modulus of numerical solution to (2.2) for $f = 1$, $\delta = 0.1$ and $\psi\restriction_{\theta=0} = 0$ and the modulus of asymptotic solution on distinctive intervals are given. All the intervals are pairwise disjoint.

Assuming $\delta$ to be small, we look for a solution $\rho$, $\alpha$ in the form of asymptotic series

$$\begin{aligned} \rho(\theta,\delta) &\sim \sum_{k=0}^{\infty} \delta^k \rho_k(\theta), \\ \alpha(\theta,\delta) &\sim \sum_{k=0}^{\infty} \delta^k \alpha_k(\theta). \end{aligned} \tag{3.2}$$

We first derive equations to determine the coefficients of these asymptotic series. For this purpose we substitute (3.2) into equations (3.1). The trigonometric functions in these equations are expanded as the Taylor series in a neighbourhood of some point $\alpha_0$. Then we equate the coefficients of the same powers of parameter $\delta$. As a result we get a recurrent sequence of triangle systems of linear equations for the unknown coefficients of (3.2). In particular, for the leading-order terms of series, we obtain

$$\begin{aligned} &\sqrt{\theta} + f\sin\alpha_0 = 0, \\ &2\theta\rho_0 - f\cos\alpha_0 = 0, \end{aligned}$$

which gives $\rho_0$ and $\alpha_0$.

On equating the coefficients of $\delta$ we arrive at the system

$$\begin{aligned} &2\alpha_1 f\sqrt{\theta}\cos\alpha_0 + 1 = 0, \\ &2\theta\rho_1 + \sqrt{\theta}\alpha_0' + \alpha_1 f\sin\alpha_0 = 0, \end{aligned}$$

which readily yields $\alpha_1$ and $\rho_1$ by

$$\alpha_1 = -\frac{1}{\sqrt{\theta}\sqrt{f^2-\theta}},$$

$$\rho_1 = \frac{1}{2\sqrt{\theta}}\left(\alpha_1 + \frac{1}{2\sqrt{\theta}\sqrt{f^2-\theta}}\right).$$

On equating the coefficients of $\delta^2$ we get the system

$$-2\alpha_2\cos\alpha_0 + \sin\alpha_0\alpha_1^2 = 0,$$

$$4\theta\rho_2 + 2\sqrt{\theta}\alpha_1' + \left(\cos\alpha_0\alpha_1^2 + 2\sin\alpha_0\alpha_2\right)f = 0,$$

implying

$$\alpha_2 = \frac{\sqrt{\theta}\alpha_1^2}{2\sqrt{f^2-\theta}},$$

$$\rho_2 = -\frac{1}{2\sqrt{\theta}}(\alpha_1' - \alpha_2) + \frac{1}{4\theta}\sqrt{f^2-\theta}\alpha_1^2.$$

On equating the coefficients of $\delta^3$, one still obtains a transparent system for two unknown functions $\alpha_3$, $\rho_3$:

$$f\cos\alpha_0\alpha_3 = -\rho_0 + \frac{f}{6}\left(\cos\alpha_0\alpha_1^3 + 6\sin\alpha_0\alpha_2\alpha_1\right),$$

$$2\sqrt{\theta}\rho_3 - \alpha_3 = -\alpha_2' - \rho_0^2 - \frac{f}{\theta}\cos\alpha_0\rho_0 + \frac{1}{6}\alpha_1^3 - \frac{f}{\sqrt{\theta}}\cos\alpha_0\alpha_2\alpha_1,$$

whose solution is

$$\alpha_3 = \frac{1}{6}\left(\alpha_1^3 - 3\theta\right) - \frac{\sqrt{\theta}}{\sqrt{f^2-\theta}}\alpha_1\alpha_2,$$

$$\rho_3 = -\frac{1}{2\sqrt{\theta}}(\alpha_2' + \alpha_3) - \frac{\sqrt{f^2-\theta}}{2\theta}\alpha_1\alpha_2 - \frac{1}{12\sqrt{\theta}}\alpha_1^3 - \frac{(\theta^2+2)(f^2-\theta)}{8\sqrt{\theta}}$$

and so on.

A careful analysis of formulae for $\alpha_k$ and $\rho_k$ obtained in this way actually shows that

$$\alpha_k = O((f^2-\theta)^{(1-2k)/2}),$$

$$\rho_k = O((f^2-\theta)^{(1-2k)/2})$$

as $\theta \to f^2 - 0$. From these equalities it follows that the constructed asymptotic expansion is valid for $\delta(f^2-\theta)^{-1} \ll 1$.

## 4. Stability of autoresonant growth

We will look for a solution which is a partial sum of the asymptotic series constructed above, up to remainders $\tilde{\rho}(\xi,\theta,\delta)$ and $\tilde{\alpha}(\xi,\theta,\delta)$. Namely, we consider

$$\psi(\theta,\delta) = (\sqrt{\theta} + \rho_1(\theta)\delta^3 + \tilde{\rho}\delta^3)\exp(\imath(\alpha_0(\theta) + \alpha_1(\theta)\delta + \tilde{\alpha}\delta^2)), \tag{4.1}$$

where $0 < \delta \ll 1$ and $\xi = \delta^{-3}\theta$ is the fast variable.

We substitute (4.1) into (2.2). The task is now to write down the linear part of the system for $\tilde{\rho}$ and $\tilde{\alpha}$. An easy computation yields the system of equations

$$\begin{aligned}\tilde{\rho}_\xi' &= \left(\sqrt{f^2-\theta}\delta - \frac{\delta^2}{2\sqrt{f^2-\theta}}\right)\tilde{\alpha} + f_1(\theta) + O(\delta^3),\\ \tilde{\alpha}_\xi' &= -2\sqrt{\theta}\tilde{\rho} + \delta\tilde{\alpha} + f_2(\theta) + O(\delta^2).\end{aligned} \tag{4.2}$$

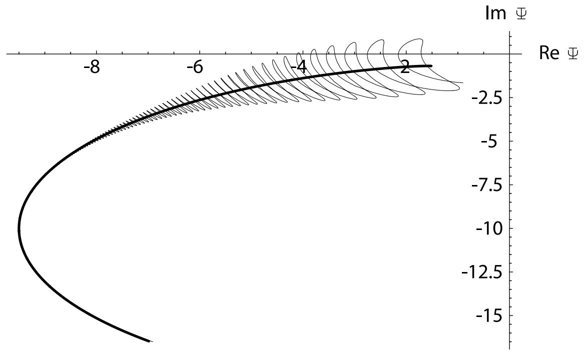


**Figure 5.** The graph displays the exponential decay of oscillations with zero initial data in a neighbourhood of a slowly varying asymptotic solution (bold curve) with $f = 1$ and $\delta = 0.05$.

The right-hand side of the system has coefficients slowly varying in the fast variable $\xi$. Solutions of such systems are usually constructed by the WKB method, see for instance [19]. The eigenvalues of the matrix on the right-hand side of (4.2) are

$$\lambda_{1,2} = \pm\imath\sqrt{2}\sqrt[4]{(f^2-\theta)\theta}\,\delta^{1/2} - \frac{\delta}{2} + O(\delta^{3/2}),$$

and so the real part of eigenvalues is negative. Hence it follows that the asymptotic solution constructed above is stable in linear approximation. Figure 5 illustrates this result.

## 5. Vicinity of the break of autoresonant growth

We change the variables by $\theta = f^2 - \delta\eta$. The new independent variable $\eta$ is stretched with respect to $\theta$. We will look for a solution of the form

$$\rho(\theta,\delta) = f + \delta r(\eta,\delta),$$
$$\alpha(\theta,\delta) = \tfrac{3}{2}\pi + \sqrt{\delta}a(\eta,\delta).$$

This substitution leads to the system for two unknown functions $r(\eta,\delta)$ and $a(\eta,\delta)$:

$$r' = -r + f(\cos(\sqrt{\delta}a) - 1)/\delta,$$
$$a' = \delta^{-5/2}(\eta - fr - \delta r^2) - \delta^{1/2}\frac{f\sin(\sqrt{\delta}a)}{f+\delta r}.$$

This system can be rewritten in a slightly different form:

$$fa^2 = 2(-r' - r - f\frac{\cos(\sqrt{\delta}a) - 1 + \delta a^2/2}{\delta},$$
$$2fr = \eta - \delta r^2 - \delta^{5/2}\left(a' - \delta^{-1/2}\frac{f\sin(\sqrt{\delta}a)}{f+\delta r}\right).$$

To find asymptotics, we substitute formal series in powers of $\sqrt{\delta}$ for $a(\eta,\delta)$ and $r(\eta,\delta)$. Namely

$$r(\eta,\delta) = \sum_{k=0}^{\infty} r_k(\eta)\delta^{k/2},$$
$$a(\eta,\delta) = \sum_{k=0}^{\infty} a_k(\eta)\delta^{k/2}.$$

On substituting these series into the system, we expand both left-hand side and right-hand side of the equalities as formal series in powers of $\sqrt{\delta}$. Then we equate the coefficients of the same powers of $\sqrt{\delta}$ in both series. As a result we arrive at a recurrent system of equations for determining the coefficients of formal series. For $k = 0$, it reads

$$2fr_0 = \eta,$$
$$fa_0^2 = -1 - \eta.$$

For $k = 1$, we get $r_1 = 0$ and $a_1 = 0$. For $k = 2$, the system is

$$2fr_2 = -r_0^2,$$
$$fa_0a_2 = -r_2' + r_2 + f\frac{a_0^4}{24},$$

implying

$$r_2 = -\frac{\eta^2}{(2f)^3},$$
$$a_2 = -\frac{2\eta^2 + 4\eta - 1}{24f^3\eta + 24f^3}\sqrt{-1-\eta}$$

and so on.

The formulae for the coefficients $r_k$ and $a_k$ are cumbersome. However, using the recurrence relations, one can see that the coefficients have a singularity at $\eta = -1$. The greater $k$, the higher the singularity. This is caused by differentiating the square root $\sqrt{-1-\eta}$ and by increasing the nonlinear dependence on lower order terms of asymptotics at each step of iteration. More precisely, we get

$$a_k = O((-1-\eta)^{(1-k)/2}),$$
$$r_k = O((-1-\eta)^{(4-k)/2})$$

as $\eta \to -1$, provided $k - 4 \in \mathbb{N}$. Hence, it follows that the constructed series is asymptotic for $\delta\,(-1-\eta)^{-1} \ll 1$.

## 6. Break of autoresonant growth

In a neighbourhood of the point $\eta = -1$, we change the variables by the formula $\eta = -1 + \tau\delta$. The new independent variable $\tau$ is fast with respect to the original variable $\eta$. The solution of the primary resonance equation is written in the form

$$a = \delta^{1/2}u(\tau, \delta),$$
$$\tau = -\frac{1}{2f} + \frac{(4f^2\tau - 1)}{8f^3}\delta + \delta^2 v(\tau, \delta).$$

Substituting these formulae into the system of equations for $a$ and $r$, we immediately obtain

$$v' - \frac{1}{\delta^2}(f\cos(\delta u) - f) + \frac{\tau}{2f} - \frac{1}{8f^3} + \delta v = 0,$$
$$u' - 2fv + \frac{1}{8f^4}(4f^2\tau - 1) + \frac{f\sin(\delta u)}{f + \delta\left(-\frac{1}{2f} + \frac{\delta}{8f^3}(4f^2\tau - 1) + \delta^2 v\right)}$$
$$-\delta\left(\frac{1}{f}v - \frac{1}{64f^6}(4f^2\tau - 1)^2\right) - \delta^2\frac{1}{4f^3}v(4f^2\tau - 1) - \delta^3 v^2 = 0.$$

We will look for a formal solution to this system in the form of power series in $\delta$:

$$u(\tau,\delta)=\sum_{k=0}^{\infty}u_k(\tau)\delta^k,\qquad v(\tau,\delta)=\sum_{k=0}^{\infty}v_k(\tau)\delta^k. \tag{6.1}$$

Substituting these series into the equations and expanding the left-hand sides as power series in $\delta$, we equate the coefficients of the same powers of $\delta$. This leads to a recurrent sequence of differential equations for $u_k$ and $v_k$. In particular, for $u_0$ and $v_0$, we get

$$u_0'+2fv_0-\frac{1}{2f^2}\left(\tau-\frac{1}{4f^2}\right)=0,$$
$$v_0'+\frac{f}{2}u_0^2+\frac{1}{2f^2}\left(\tau-\frac{1}{4f^2}\right)=0.$$

For $u_1$ and $v_1$, the system looks like

$$u_1'+2fv_1=\frac{1}{f}v_0-u_0-\frac{1}{4f^2}\left(\tau-\frac{1}{4f^2}\right)^2,$$
$$v_1'+\frac{f}{2}u_0u_1=-v_0.$$

For $u_2$ and $v_2$, the system is

$$u_2'+2fv_2=\frac{1}{f}v_1-\frac{1}{4f^3}(1-4f^2\tau)v_0-u_1-\frac{1}{2f^2}u_0,$$
$$v_2'+\frac{f}{2}u_0u_2=-v_1+\frac{f}{24}u_0^4-\frac{f}{2}u_1^2$$

and so on.

The system of equations for the leading-order terms reduces to the Painlevé-1 equation. To see this, let

$$u_0=\sqrt[3]{\frac{f^2}{6}}y(z,c_1,c_2),$$
$$v_0=\sqrt[3]{\frac{9}{2f^7}}\frac{\mathrm{d}}{\mathrm{d}\tau}(y(z,c_1,c_2))+\frac{z}{24f}-\frac{1}{8f^7},$$
$$\tau+\frac{1}{4f^2}=\frac{f^2}{6}z.$$

Then, the differentiation of the equation for $u_0$ leads, by the second equation, to the Painlevé-1 equations in the standard form $y''=6y^2+z$. Here $y=y(z,c_1,c_2)$ is the first Painlevé transcendental, $c_1$ and $c_2$ are real parameters of the transcendental which are monodromy data, cf [18], and $z$ is an independent variable.

The solution of the system of equations for the leading-order term is determined through the first Painlevé transcendental. The parameters of the transcendental are defined by making asymptotic expansions consistent. To this end we re-expand asymptotic series (6.1) in terms of the variable $\tau$ and equate the coefficients of the same powers of $\delta$. Then we get asymptotics of the coefficients for $\tau\to-\infty$, namely

$$u_0=-\frac{1}{f}\sqrt{-\tau}+O\left(\frac{1}{\sqrt{-\tau}}\right),$$
$$v_0=\frac{1}{4f^3}\tau-\frac{1}{16f^5}+O\left(\frac{1}{\sqrt{-\tau}}\right);$$

$$u_1 = \frac{1}{8f^5}\left(\frac{1}{\sqrt{-\tau}}\right) + O\left(\frac{1}{\tau}\right),$$
$$v_1 = \frac{1}{8f^3}\tau^2 - \frac{3}{16f^3}\tau + O(\sqrt{-\tau});$$

and

$$u_2 = \frac{1}{12f^3}\sqrt{-\tau^3} + \frac{5}{32f^5}\sqrt{-\tau} + \frac{1}{2f^2} + O\left(\frac{1}{\sqrt{-\tau}}\right),$$
$$v_2 = \frac{3}{16f^5}\tau^2 - \frac{5}{32f^7}\tau + O(\sqrt{-\tau}).$$

The asymptotics of the $k$ th correction is

$$u_{2n} = O(\sqrt{-\tau^{2n+1}}), \qquad u_{2n+1} = O(\sqrt{-\tau^{2n-1}}),$$
$$v_{2n} = O(\tau^{n+1}), \qquad v_{2n+1} = O(\tau^{n+2})$$

as $\tau \to -\infty$.

The solution $u_0$ and $v_0$ with given asymptotics as $\tau \to -\infty$ can be expressed through the first Painlevé transcendental. The asymptotics of the first Painlevé transcendental were investigated in [18, 20, 21]. Here it is convenient to make use of the connection of the asymptotics and the monodromy data:

$$u_0 = \sqrt[3]{\frac{f^2}{6}}\, y(z, c_1, c_2)\restriction_{\substack{c_1=0\\ c_2=0}},$$

see [18]. Starting with the formula for $u_0$, one obtains easily an expression for $v_0(\tau)$ from the first equation of the system for $u_0$ and $v_0$.

We now turn to construction of solutions $u_k$ and $v_k$. The corresponding homogeneous system is

$$U' + 2fV = 0,$$
$$V' + \frac{f}{2}u_0 U = 0. \tag{6.2}$$

Set

$$U(\tau) = \sqrt[3]{\frac{f^2}{6}}\, w(z).$$

On differentiating the first equation and substituting $V'$ into the second equation, we arrive at the linearized Painlevé-1 equation $w'' + 2u_0 w = 0$. The general solution of this equation is known to be a linear combination of the partial derivatives of the first Painlevé transcendental in parameters, i.e.

$$w = A_1\partial_{c_1} y(z, c_1, c_2) + A_2\partial_{c_2} y(z, c_1, c_2),$$

where $A_1$ and $A_2$ are arbitrary constants. The asymptotics of $y(z, c_1, c_2)$ as $z \to -\infty$ implies

$$\partial_{c_1} y(z, c_1, c_2)\restriction_{\substack{c_1=0\\ c_2=0}} = O(z^{-5/8}),$$
$$\partial_{c_2} y(z, c_1, c_2)\restriction_{\substack{c_1=0\\ c_2=0}} = O(z^{3/8}),$$

see [18].

The formulae for the corrections $u_k$ and $v_k$ can now be obtained by the method of variation of constants:

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \Phi(\tau)\begin{pmatrix} A_k \\ B_k \end{pmatrix} + \Phi(\tau)\int_a^\tau \Phi(\tau')^{-1}\begin{pmatrix} f_k \\ g_k \end{pmatrix} \mathrm{d}\tau'.$$

Here $\Phi(\tau)$ stands for the fundamental matrix of the linearized system (6.2) with Wronskian equal to 1. By $a$ is meant an arbitrary real constant satisfying

$$a < \tau_0 := (f^2/6)z_0 - 1/(4f^2),$$

where $z_0 \sim 2.38$ is the least real pole of the Painlevé transcendental. The constants $A_n$ and $B_n$ are uniquely determined from the matching condition for asymptotic solutions.

The first Painlevé transcendental has second-order poles on the real axis, see [21]. In a neighbourhood of $\tau_0$, the constructed asymptotic expansion no longer holds. Indeed, we get

$$u_0 \sim \frac{6}{f^2(\tau-\tau_0)^2},$$
$$v_0 \sim \frac{6}{f^3(\tau-\tau_0)^3}$$

for $\tau$ close to $\tau_0$.

The general solution for the first correction $u_1$ and $v_1$ can be represented in the form

$$u_1 = \frac{a_1}{(\tau-\tau_0)^3} + \frac{3}{f^4(\tau-\tau_0)^2} + O((\tau-\tau_0)^{-1}),$$
$$v_1 = \frac{3a_1}{2(\tau-\tau_0)^4} + \frac{6}{f^5(\tau-\tau_0)^3} + O((\tau-\tau_0)^{-2}).$$

Here $a_1$ is one of the solution parameters. The second independent parameter is contained in the smooth part of asymptotics remainder. The parameters of the solution are uniquely determined while one constructs it by the method of variation of constants. However, in the expansion in a neighbourhood of the pole $\tau_0$, the parameter $a_1$ can be included in the pole translation of the leading-order term of order $\delta$, namely $\tau_1 = \tau_0 - \delta f^2 a_1/12$. Computations show that $a_1 \sim -38.25$. As a result the value $\tau_0$ in the expansion of leading-order terms should be replaced by $\tau_1$ and the expansions for $u_1$ and $v_1$ become

$$u_1 = \frac{3}{f^4(\tau-\tau_1)^2} + O((\tau-\tau_1)^{-1}),$$
$$v_1 = \frac{6}{f^5(\tau-\tau_1)^3} + O((\tau-\tau_1)^{-2}).$$

Thus, the pole of the leading-order term of asymptotics is defined uniquely up to $\delta^2$. More precisely, the pole asymptotics of the perturbed problem is determined by singling out summands of order $(\tau-\tau_1)^{-3}$ in the asymptotics of $u_k$ for $k > 1$.

The order of singularity at the point $\tau = \tau_1$ increases, for the higher order corrections depend on lower order corrections in a nonlinear way. For $u_2$ and $v_2$, we have

$$u_2 \sim \frac{-18}{5f^6(\tau-\tau_1)^6},$$
$$v_2 \sim \frac{-54}{5f^7(\tau-\tau_1)^7}.$$

One can show that

$$u_{2n-1} = O((\tau-\tau_1)^{-2n}), \qquad u_{2n} = O((\tau-\tau_1)^{-4n-2}),$$
$$v_{2n-1} = O((\tau-\tau_1)^{-2n-1}), \qquad v_{2n} = O((\tau-\tau_1)^{-4n-3})$$

as $\tau \to \tau_1$.

From the behaviour of $u_k$ and $v_k$ in a neighbourhood of singular point, we deduce that the constructed asymptotics is valid in the domain

$$\frac{\sqrt{\delta}}{|\tau-\tau_1|} \ll 1.$$

## 7. Fast motion

In a neighbourhood of the singular point $\tau_1$, the behaviour of the solution changes drastically. The solution begins to vary quickly. The new scale of independent variable is now

$$\xi = \frac{(\tau - \tau_1)}{\sqrt{\delta}}.$$

One introduces new dependent variables $p(\xi, \delta)$ and $s(\xi, \delta)$ by

$$\begin{aligned} \rho &= f - 2\frac{\delta}{f} + \delta\sqrt{\delta}\, p(\xi, \delta), \\ \alpha &= \frac{3}{2}\pi + s(\xi, \delta). \end{aligned} \tag{7.1}$$

The genuine independent variable $\theta$ is related to the new independent variable $\xi$ by the formula

$$\theta = f^2 - \delta + \delta^2 \tau_1 + \sqrt{\delta^5}\xi.$$

Substituting the expressions for $\rho$, $\alpha$ and $\theta$ into the original system (3.1) yields a system of equations for $p(\xi, \delta$ and $s(\xi, \delta)$. This system is cumbersome and we need not write it here in an explicit form. Using the standard procedure of perturbation theory, we look for the leading-order term of asymptotics in $\delta$ of the form

$$\begin{aligned} p(\xi, \delta) &\sim p_0(\xi), \\ s(\xi, \delta) &\sim s_0(\xi). \end{aligned}$$

For $p_0(\xi)$ and $s_0(\xi)$ we obtain the system

$$\begin{aligned} p_0' + f(1 - \cos s_0) &= 0, \\ s_0' + 2 f p_0 &= 0. \end{aligned}$$

This system admits the conservation low

$$E_0 = p_0^2 + (\sin s_0 - s_0).$$

Making (7.1) and asymptotics in a neighbourhood of the pole consistent gives a condition for $p_0$ and $s_0$, namely both $E_0$ and $s_0$ vanish as $\eta \to -\infty$. The solution of the system for $p_0$ and $s_0$ varies quickly and $s_0$ increases infinitely. The variable $s_0$ stands actually for the argument of the solution in complex form

$$\left( f - 2\frac{\delta}{f} + \delta\sqrt{\delta} p(\xi, \delta) \right) \exp\left( \imath \frac{3}{2}\pi + s(\xi, \delta) \right).$$

Thus, in this mode, the phase-locking condition fails to hold and the solution is not autoresonant.

## 8. Conclusion

In the paper, it is shown for Duffing's equation that in the autoresonance domain, there exists an asymptotic solution which is stable in linear approximation. We evaluate the break time and the maximal amplitude of the autoresonant solution for the equation with small dissipation. The break of the autoresonant mode is accompanied by the hard loss of stability and passage to fast oscillations.

## Acknowledgments

The authors are greatly indebted to L Kalyakin for many stimulating conversations. They gratefully acknowledge the many helpful suggestions of L Friedland, M Shamsutdinov and A Sukhonosov. The research was supported by the RFBR grant 09-01-92436-KE-a, the DFG grant TA 289/4-1 and grant 2215.2008.1 for Russian scientific schools. Analytic calculations were partially performed by means of the program GNU Maxima (http://maxima.sourceforge.net) under GNU TeXmax (http://www.texmacs.org/). The authors also wish to express their thanks to elaborators of these free software tools.

## References

[1] Veksler V I 1945 A new method of acceleration of relativistic particles *J. Phys. USSR* **9** 153–8
[2] McMillan E M 1945 The synchrotron—a proposed high energy particle accelerator *Phys. Rev.* **68** 143–4
[3] Fajans J and Friedland L 2001 Autoresonant (non-stationary) excitation of pendulums, Plutinos, plasmas, and other nonlinear oscillators *Am. J. Phys.* **69** 1096–102
[4] Kalyakin L A 2008 Asymptotic analysis of autoresonance models *Russ. Math. Surv.* **63** 791–857
[5] Yariv S and Friedland L 1993 Autoresonance interaction of three nonlinear adiabatic oscillators *Phys. Rev.* E **48** 3072–6
[6] Khain E and Meerson B 2001 Parametric autoresonance *Phys. Rev.* E **64** 036619
[7] Fajans J, Gilson E and Friedland L 2001 The effect of dumping on autoresonant (nonstationary) excitation *Phys. Plasmas* **8** 243–7
[8] Naaman O, Aumentado J, Friedland L, Wurtele J S and Siddiqi I 2008 Phase-locking transition in a chirped superconducting Josephson resonator *Phys. Rev. Lett.* **101** 117005
[9] Shagalov A G, Juul Rasmussen J and Naulin V 2009 Phase-locking phenomena and excitation of damped and driven nonlinear oscillators *J. Phys. A: Math. Theor.* **42** 045502
[10] Shamsutdinov M A, Kalyakin L A, Khalfina A A and Sukhonosov A L 2008 Autoresonance in a ferromagnetic plate with a stripe domain structure *Izv. Ross. Akad. Nauk, Ser. Fiz.* **72** 1487–9
[11] Shamsutdinov M A, Kalyakin L A, Sukhonosov A L and Khalfina A A 2009 Autoresonant excitation of oscillations of domain walls in a ferromagnetic film *Phys. Met. Metallogr.* **108** 8–18 (Engl. Transl.)
Shamsutdinov M A, Kalyakin L A, Sukhonosov A L and Khalfina A A 2009 *Fiz. Met. Metallcved.* **108** 10–21
[12] Yaakobi O, Friedland L, Henis Z and Driven 2007 Autoresonant three-oscillator interaction *Phys. Rew.* E **76** 026205
[13] Sinclair A T 1972 On the origin of the commensurabilities amongst the satellites of Saturn *Mon. Not. R. Astron. Soc.* **160** 169–87
[14] Bogolyubov N N and Mitropol'skij Yu A 1961 *Asymptotic Methods in the Theory of Nonlinear Oscillations* (New York: Gordon and Breach)
[15] Haberman R 1977 Nonlinear transition layers—the second Painlevé transcendent *Stud. Appl. Math.* **57** 247–70
[16] Kiselev O M and Glebov S G 2003 An asymptotic solution slowly crossing the separatrix near a saddle-center bifurcation point *Nonlinearity* **16** 327–62
[17] Kiselev O M and Glebov S G 2007 The capture into parametric autoresonance *Nonlinear Dyn.* **48** 217–30
[18] Kapaev A A 1988 Asymptotics of solutions of the Painlevé equations of the first kind *Diff. Eq.* **24** 1684–93 (Engl. Transl.)
[19] Wasow W 1965 *Asymptotic Expansions for Ordinary Differential Equations* (New York: Wiley)
[20] Holmes P and Spence D 1984 On a Painlevé-type boundary value problem *Q. J. Mech. Appl. Math.* **37** 525–38
[21] Gromak V I and Lukashevich N A 1990 *Analytical Properties of Solutions of the Painlevé Equations* (Minsk: Universitetskoe)